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$(n-1)$-AXIAL $SO(n)$ AND $SU(n)$ ACTIONS ON HOMOTOPY SPHERES

BY

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ABSTRACT. Let $G(n) = O(n)$ or $U(n)$ and $SG(n) = SO(n)$ or $SU(n)$. For each integer $m \geqslant 1$ a family $\{S_{\gamma,\sigma}: \gamma \in H, \sigma \in K\}$ of $(n-1)$-axial $SG(n)$ homotopy spheres $S_{\gamma,\sigma}$ is constructed. Each $S_{\gamma,\sigma}$ has fixed point set of dimension $(m-1) \geqslant 0$ and orbit space of dimension $r = \frac{1}{2}n(n-1) + (m-1)$ (resp. $r = (n-1)^2 + m - 1$) if $SG(n) = SO(n)$ (resp. $SU(n)$). H is $\pi_{r-1}(SG(n)/G(n-1))$. K is trivial if $SG(n) = SO(n)$ and is a homotopy theoretically defined subgroup of sections of an S^2 bundle depending only on m and n if $SG(n) = SU(n)$. Assume that m and n satisfy the mild restriction §5, (1). It is shown that the above family is universal for $(n-1)$-axial $SG(n)$ homotopy spheres and provides a classification analogous to the classification of fibre bundles: for each $(n-1)$-axial $SG(n)$ homotopy sphere Σ there is a $S_{\gamma,\sigma}$ and a unique equivariant stratified map $\Sigma \to S_{\gamma,\sigma}$. Σ is equivariantly diffeomorphic to the pullback of $S_{\gamma,\sigma}$ via the map $B(\Sigma) \to B(S_{\gamma,\sigma})$ of orbit spaces. If $SG(n) = SO(n)$ then γ is unique (and $\sigma = 1$). If $SG(n) = SU(n)$ then γ is unique modulo the image of

$$\pi_{r-1}S(U(n-2) \times U(2))/U(k-1) \times U(1) \quad \text{in } H.$$

An example is given showing that the differentiable structure of the underlying smooth manifold of $S_{\gamma,\sigma}$ may be exotic.

0. Introduction. For an excellent introduction to multiaxial actions the reader is referred to Davis' book [7]. Our notation will conform more to his thesis [**6**]; in particular, we use the definition given there for twist invariants.

Let $G(n) = O(n)$ or $U(n)$ and $SG(n) = SO(n)$ or $SU(n)$ except where otherwise specified. If $G = G(n)$ or $SG(n)$, then $k\rho_n$ denotes k times the standard representation of G, i.e., the natural action (by matrix multiplication) of G on the space $M(n, k)$ of $n \times k$ matrices with coefficients in $\mathbf{R}$ if $G = O(n)$ or $SO(n)$, or $\mathbf{C}$ if $G = U(n)$ or $SU(n)$. A smooth $G(n)$ or $SG(n)$ action on a smooth manifold M is called k-axial if the normal representation at each point is $(k-i)\rho_{n-i}$ for some i. M is a stratified space, its ith stratum consists of points $x \in M$ whose normal representation is $(k-i)\rho_{n-i}$, or equivalently $G(n)_x \sim G(n-i)$ (or $SG(n)_x \sim SG(n-i)$). The orbit space of $M(n,k)$ is $H_+(k)$, where $H_+(k)$ denotes the space of positive semidefinite symmetric (resp. Hermitian) matrices if $G = O(n)$ or $SO(n)$ (resp. $U(n)$ or $SU(n)$). Let M_i and $N_i(M)$ (or simply N_i) denote the closed ith stratum and closed ith normal bundle of M defined inductively by letting M_0 equal the zeroth stratum of M, letting N_0 equal a closed equivariant tubular neighborhood of M_0, and letting M_i and N_i equal the ith stratum and ith normal bundle respectively of $\mathrm{cl}(M - \bigcup_{j<i} N_j)$. Let $B(M)$ (or simply B) be the orbit space of M.

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Let $B_i(M) = M_i/G$ and $C_i(M) = N_i/G$. The composite projection $N_i \to M_i \to B_i$ is a fibre bundle called the ith normal orbit bundle. Let P_i denote the associated principal bundle and S_i its structure group. P_i and S_i are called the ith principal orbit bundle and the ith structure group, respectively. If $(X, \| \|)$ is a normed space, define $D(X) = \{x \in X: \|x\| \leqslant 1\}$ and $S(X) = \{x \in X: \|x\| = 1\}$. The projection $C_i = N_i/G \to B_i$ is called the ith normal cone bundle. Its fibre is isomorphic to $D(H_+(k-i))$, where the norm on $H_+(k-i)$ is given by the sum of the squares of the moduli of eigenvectors of a matrix. Note that M is a union of codimension zero pieces N_i, and $B(M)$ is a union of codimension zero pieces C_i which intersect in codimension-1 subspaces of $B(M)$. $B(M)$ is a stratified space; each stratum $B_i(M)$ inherits the structure of a smooth manifold from M. $B(M)$ is not a smooth manifold but has a smooth functional structure (see [**1**]) induced from M. An equivariant stratified map F: $M \to M'$ of k-axial G manifolds is a smooth strata preserving map which induces bundle maps $N_i(M) \to N_i(M')$. (Such maps are essentially the same as the transverse linear isovariant maps of Browder and Quinn [**5**].) A stratified map $F_\#$: $B(M) \to B(M')$ is a strata preserving map which induces bundle maps $C_i(M) \to C_i(M')$ and which is "smooth", i.e., if f: $B(M') \to \mathbf{R}$ is a member of the functional structure of $B(M')$, then $f \circ F_\#$ is a member of the functional structure of $B(M)$.

Let $SO(3)$ and $O(3)$ act biaxially on $M(3,2) \times \mathbf{R}^2$ by left matrix multiplication on the first factor. Let S^7 be the unit sphere in $M(3,2) \times \mathbf{R}^2$ (in some $SO(3)$ invariant metric). Let Σ^7 be the generator of the group of homotopy seven spheres.

This work was motivated by the example of Gromoll and Meyer [**14**] of a biaxial $SO(3)$ action on Σ^7. As remarked in [**7**] an $SO(3)$ equivariant stratified map F: $\Sigma^7 \to S^7$ cannot exist. (Such an F would induce a biaxial $O(3)$ action on Σ^7. But $B(\Sigma^7) = B(S^7)$ and biaxial $O(3)$ actions are determined up to equivariant diffeomorphism by their orbit spaces. This would imply Σ^7 is diffeomorphic to S^7, a contradiction.)

Let $G(n) = O(n)$ or $U(n)$ and $SG(n) = SO(n)$ or $SU(n)$. The general theory of Davis [**6, 7**] of k-axial $G(n)$ actions, where $k \leqslant n$, implies that such actions pull back from the linear model $k\rho_n = M(n, k)$ and that such actions on homotopy spheres with fixed point sets of dimension $m-1$ pull back from the linear sphere S^{nk+m-1} ($=$ unit sphere in $M(n,k) \times \mathbf{R}^m$). By the example of the previous paragraph (where $k = 2 = n-1$, $m = 2$), this theory does not apply to k-axial $SG(n)$ actions if $k = n-1$. In fact, if $k < n-1$, then k-axial $SG(n)$ actions correspond uniquely to k-axial $G(n)$ actions (see §2). Thus $k = n-1$ is the first interesting case.

We observe that except for the top two strata, $(n-1)$-axial $SG(n)$ actions correspond to $(n-1)$-axial $G(n)$ actions and develop an obstruction theory for extending $(n-1)$-axial $SG(n)$ actions to $G(n)$ actions. In (3.1) we show that the obstructions to extending $(n-1)$-axial $SO(n)$ actions on homotopy spheres with fixed points to $(n-1)$-axial $O(n)$ actions over $\mathrm{cl}(M - N_k(M))$ lie in trivial groups (except for biaxial $SO(3)$ actions on S^6 which can be classified by other methods). The corresponding obstructions for $SU(n)$ lie in many nontrivial groups; however, for homotopy spheres the complete obstruction to extending the action to $G(n)$ over

$\mathrm{cl}(M - N_k(M))$ can be measured as a homotopy class of a section of a certain S^2 bundle. The S^2 bundle depends only on the total dimension of the homotopy sphere and the dimension of its fixed point set. The set of homotopy classes of sections which can be realized as the complete obstruction to extending the action to $G(n)$ over $\mathrm{cl}(M - N_k(M))$ is the homotopy theoretically defined group K of (3.2). In §4 we analyze the obstruction $\gamma \in \pi_{r-1}(SG(n)/G(n-1))$ to completing the extension over the top stratum. The set H of such γ which can be realized is all of $\pi_{r-1}(SG(n)/G(n-1))$.

The constructions realizing the obstructions yield a universal family $\{S_{\gamma,\sigma}: \gamma \in H, \sigma \in K\}$ of $(n-1)$-axial $SG(n)$ homotopy spheres with fixed point sets of dimension $(m-1) \geqslant 0$. For technical reasons we assume $(m-1) \geqslant 2$ if $n = 4$ and $(m-1) \geqslant 1$ if $n = 5$ for $SG(n) = SO(n)$ and $(m-1) \geqslant 2$ if $n = 3$ for $SG(n) = SU(n)$. (This is because we apply the surgery results of Davis and Hsiang [**11, 13**].)

§5 consists of the statement and proof of our main theorem. We show that the universal family has the property that each $(n-1)$-axial $SG(n)$ homotopy sphere with fixed point set of dimension $(m-1) \geqslant 0$ (with the mild restriction of the previous paragraph on $(m-1)$) admits a unique equivariant stratified map $\Sigma \to S_{\gamma,\sigma}$ for some $(\gamma, \sigma) \in H \times K$, where (γ, σ) is unique (and $\sigma = 1$) if $SG(n) = SO(n)$ and γ is unique modulo $\mathrm{Im}\, r$ (see Lemma 3.7), and σ is unique if $SG(n) = SU(n)$. This yields a one-to-one correspondence between $(n-1)$-axial $SU(n)$ homotopy spheres over a local orbit space B and stratified maps $B \to B(S_{\gamma,\sigma})$ satisfying appropriate homological conditions (see Theorem 5.1).

The spaces Σ_γ, constructed in §4 are interesting in their own right. Their differential structures are not all standard; for example, if $SG(n) = SO(n)$, $n = 3$, $m = 2$ and γ is a generator of $\pi_3(SO(3)/O(2)) \cong \mathbf{Z}$, then Σ_γ generates the group of homotopy seven spheres. In §6 we study the differential structure of Σ_γ and show that if

$$\gamma \in \mathrm{Im}(p_\#: \pi_{r-1}SG(n) \to \pi_{r-1}(SG(n)/G(n-1))),$$

then $\Sigma_\gamma \in bP_{n+1}$. If, in addition, $SG(n) = SO(n)$, $n \geqslant 4$ and $m - 1 \geqslant 2$, or $SG(n) = SU(n)$, $n \geqslant 3$ and $m \geqslant 2n$, then we show Σ_γ has standard differential structure.

In §7 we compare our results for biaxial $SO(3)$ actions with those of [**10**]. We note that answers to the conjectures raised on p. 15 of [**10**] are implicit in our main theorem.

This paper consists of the author's thesis written under the direction of Professor Wu Chung Hsiang together with work done in 1981/1982 at Purdue University. I would like to thank Professor Hsiang for his guidance and encouragement and for suggesting the study of multiaxial actions. I am also grateful to Michael Davis for several useful conversations, and for acting as my advisor while Professor Hsiang was at Stanford at the end of the 1980 spring semester.

1. Preliminaries. On first reading some may wish to omit the material following this paragraph up to Theorem 1.1. However, Definitions 1.1 and 1.2 of normal systems and twist invariants and the calculations (18) and (19) of this section play a crucial role. These calculations show that the $SG(n)$ normal systems indeed characterize the manifolds with $(n-1)$-axial $SG(n)$ action. As a general guide to the notation, ∂ refers to boundary operators on the orbit space. Since each stratum is a manifold with faces, one has operators ∂_j which restrict to the jth stratum of the boundary of the ith stratum. If j is larger than i, then $\partial_j B_i$, $\partial_j C_i$ refer to the parts of the normal bundle of the ith stratum which meet the jth stratum. This convention gives the symmetry $\partial_j N_i = \partial_i N_j$, $\partial_j C_i = \partial_i C_j$ of (1)–(4) below.

Define

(1) $$\partial_j N_i = N_i \cap N_j,$$

(2) $$\partial_{jl} N_i = N_i \cap N_j \cap N_l,$$

(3) $$\partial_j B_i = \begin{cases} C_i \cap C_j & \text{if } j > i, \\ B_i \cap C_j & \text{if } j < i, \end{cases}$$

(4) $$\partial_{jl} B_i = \partial_j B_i \cap \partial_l B_i \quad \text{if } j, l < i.$$

Note that if $j > i$, $\partial_j B_i$ is a subset of C_i and the projection $\partial_j B_i \to B_i$ is a bundle projection with fibre $S(H_+(k-i))_{j-i}$ which is homotopy equivalent to $G(k-i)/G(j-i) \times G(k-j)$.

Define

(5) $$\partial_j P_i = P_i|_{\partial_j B_i},$$

(6) $$\partial_{jl} P_i = P_i|_{\partial_{jl} B_i}.$$

Now $\partial_{i+j} P_i(M)$ is the associated principal bundle to $\partial_{i+j} N_i(M)$, and $\partial_i P_{i+j}(M)$ is the associated principal bundle to $\partial_i N_{i+j}(M)$. Thus there is a natural map

(7) $$f^{ij}\colon \partial_{i+j} P_i(M) \to \partial_i P_{i+j}(M)$$

induced by the equality

(8) $$\partial_{i+j} N_i \xrightarrow{=} \partial_i N_{i+j}.$$

In fact, the structure group of $\partial_{i+j} N_i$ reduces further, e.g., to $O(i) \times O(j) \times O(k-i-j)$ which is contained in $O(i+j) \times O(k-i-j)$ in the $O(n)$ case, and in this case the map

(9) $$\times f^{ij}\colon O(i+j) \times_{O(i)\times O(j)} \partial_{i+j} P_i(M) \cong \partial_i P_{i+j}(M),$$

defined by

(10) $$\times f^{ij}(g, x) = g f^{ij}(x),$$

is a bundle isomorphism, where $O(i+j)$ acts on $\partial_i P_{i+j}(M)$ as a subgroup of the structure group $O(i+j) \times O(k-i-j)$ of $\partial_i P_{i+j}(M)$. Let P^j_{k-i} denote the jth principal orbit bundle $P_j(M(n-i, k-i))$ of the linear model $M(n-i, k-i)$. Let sP^j_{k-i} denote the corresponding bundle for the unit sphere $S(M(n-i, k-i))$ of the linear model. Henceforth we denote $\partial_{i+j} P_i$ by the P^{ij}. The properties of f^{ij} for $G(n)$ and $SG(n)$ will be evident from what follows.

A k-axial $G(n)$ or $SG(n)$ manifold is uniquely determined by its normal system defined below.

Let $G = G(n)$ or $SG(n)$.

DEFINITION 1.1. A k-axial normal system consists of the following data:

(i) i-manifolds B_i for $0 \leqslant i \leqslant k$,

(ii) Principal S_i bundles $P_i \to B_i$, where

$$S_i = G(i) \times G(k-i) \quad \text{if } G = G(n) \tag{11}$$

and

$$S_i = \begin{cases} G(i) \times G(k-i), & k \leqslant n-2 \text{ or } i \leqslant k-2, \\ S(G(i) \times G(2)), & k = n-1 \text{ and } i = k-1, \\ SG(n), & k = n, \end{cases} \tag{12}$$

if $G = SG(n)$. (For a calculation of S_i see Davis [**8**].)

(iii) For $i < i + j \leqslant k$, $G(i) \times G(j) \times G(k-i-j)$ equivariant inclusions

$$f^{ij}\colon P^{ij} \to \partial_i P_{i+j} \quad \text{for} \begin{cases} G = G(n), & \\ G = SG(n), & 0 \leqslant i < k,\ k \leqslant n-2, \\ G = SG(n), & 0 \leqslant i < k-1,\ k = n-1, \end{cases} \tag{13}$$

and a $G(j) \times G(k-i-j) \times_{G(k-i)} S(G(i) \times G(2))$ equivariant inclusion

$$f^{ij}\colon P^{ij} \to \partial_i P_{i+j} \quad \text{if } k = n-1,\ i = k-1 \text{ and } G = SG(n). \tag{14}$$

If S_{ij} denotes the structure group of P^{ij}, then

$$\times f^{ij}\colon S_{i+j} \times_{S_{ij}} P^{ij} \to \partial_i P_{i+j} \tag{15}$$

is required to be a diffeomorphism.

(iv) For $i < i + j < i + j + l \leqslant k$, the diagram

$$\begin{array}{ccccc} P^{ijl} & \xrightarrow{\times f^{jl}} & \partial_{i+j} P^{i,j+l} & \xrightarrow{f^{i,j+l}} & \partial_i \partial_{i+j} P_{i+j+l} \\ \Big\downarrow{\scriptstyle \times f^{ij}} & & & \nearrow{\scriptstyle f^{i+j,l}} & \\ \partial_i P^{i+j,l} & & & & \end{array} \tag{16}$$

is commutative, where

$$P^{ijl} = \partial s P^{l}_{k-i-j} \times_{G(k-i-j)} \partial s P^{j}_{k-i} \times_{G(k-i)} P_i. \tag{17}$$

REMARK. Given a k-axial G manifold M, construction of the data of a normal system (N_i, P_i, f^{ij}, etc.) is evident from the previous discussion. The definition of P^{jl} when $G = SG(n)$ is justified by the calculation below which shows that $\partial_{j+l} N_j$ is an associated bundle to P^{jl}.

Let $D(n-i, k-i)$ be the unit disc in $M(n-i, k-i)$, where the norm of $x \in M(n-i, k-i)$ is given by the sum of the moduli of eigenvectors of x.

(18)

$$\begin{aligned}
\partial_{j+l} N_j &= \partial_{j+l}\big(SG(n) \times_{SG(n-j)} D(n-j, k-j)\big) \times_{S_j} P_j \\
&= \big(SG(n) \times_{SG(n-j)} \partial_j D(n-j, k-j)\big) \times_{S_j} P_j \\
&= \big\{ SG(n) \times_{SG(n-j)} \big(SG(n-j) \times_{SG(n-l-j)} D(n-l-j, k-l-j) \\
&\qquad\qquad \times_{G(k-l-j)\times G(l)} \partial s P^l_{k-j}\big)\big\} \times_{S_j} P_j \\
&= \big\{ SG(n) \times_{SG(n-l-j)} D(n-l-j, k-l-j)\big\} \\
&\qquad \times_{G(k-l-j)\times G(l) \times G(j)} \big\{ \partial s P^l_{k-j} \times_{G(k-j)} P_j \big\} \\
&= \big\{ SG(n) \times_{SG(n-l-j)} D(n-l-j, k-l-j)\big\} \times_{G(k-l-j)\times G(l)\times G(j)} P^{jl}
\end{aligned}$$

if $j < k-1$.

If $j = k-1$, then $l = 1$ and

$$\begin{aligned}
\partial_k N_{k-1} = \partial_{j+l} N_j &= SG(n) \times_{G(l)} \partial s P^1_1 \times_{S_{k-1}} P_{k-1} \\
&= SG(n) \times_{S_{k-1}} \big(\partial s P^1_1 \times_{G(1)} P_{k-1}\big),
\end{aligned} \tag{19}$$

where $G(1)$ acts on P_{k-1} by the inclusion

$$G(1) \hookrightarrow S(G(k-1) \times G(1) \times G(1)) \to S(G(k-1) \times G(2)) = S_{k-1} \tag{20}$$

which sends $G(1)$ to the middle factor.

In either case

$$\partial_{j+l} N_j = \big(SG(n) \times_{SG(n-l-j)} D(n-l-j, k-l-j)\big) \times_{S_{jl}} P^{jl}. \tag{21}$$

Let M be a k-axial $G(n)$ manifold with $k \leqslant n$ or a k-axial $SG(n)$ manifold with $k \leqslant n-1$. Suppose that the bundle of principal orbits $P_k(M)$ is trivial and let T: $P_k(M) \to S_k$ be a trivialization. Recall that S_i denotes the structure group of $P_i(M)$, so that S_k is the principal isotropy subgroup.

Let $sH_+(k-i)$ denote the set $\{x \in H_+(k-i): \|x\| = 1\}$. We will define a map

$$I: P_i(M) \to P^{i,k-i}(M). \tag{22}$$

This map serves to push the ith stratum into the bundle of principal orbits of a tubular neighborhood. Recall that

$$P^{i,k-i}(M) \cong sP^{k-i}_{k-i} \times_{G(k-i)} P_i(M) \tag{23}$$

and

$$sP^{k-i}_{k-i} \cong sH_+(k-i) \times G(k-i). \tag{24}$$

Let $y \in sH_+(k-i)$ be fixed under the action of $G(k-i)$ on $sH_+(k-i)$ by conjugation.

Let $c \in sP_{k-i}^{k-i}$ correspond to (y, h) under the identification (24). Define

$$I(x) = [c, x]. \tag{25}$$

DEFINITION 1.2. The ith upper twist invariant $\tilde{f}_i(M)$ is given by the composite

$$P_i(M) \xrightarrow{I} sP_{k-i}^{k-i} \times_{G(k-i)} P_i(M) \xrightarrow{f^{i,k-i}} \partial_i P_k(M) \xrightarrow{T} S_k. \tag{26}$$

The ith twist invariant $f_i(M)$ is defined by the induced map

$$B_i(M) \cong P_i(M)/S_i \to S_k/S_i \tag{27}$$

on the orbit space level.

Note. The choice of c above guarantees that $f_i(M)$ is well defined.

THEOREM 1.1 (DAVIS [6, p. 81]). *Let M be a k-axial $G(n)$ manifold, $G(n) = O(n)$ or $U(n)$, and suppose M has trivial bundle of principal orbits. Let T: $P_k(M) \to G(k)$ be a trivialization. Then there exists an equivariant stratified map F: $M \to M(n, k)$ from M to the linear model which preserves the trivializations, i.e.*

$$\begin{array}{ccccc} P_k(F): P_k(M) & \to & P_k(M(n,k)) & \cong & H_+(k)^k \times G(k) \\ & T \searrow & & \swarrow \text{proj}_2 & \\ & & G(k) & & \end{array} \tag{28}$$

is a commutative diagram. F is unique up to isotopy through equivariant stratified maps. M is equivariantly diffeomorphic to the pullback of π: $M(n, k) \to H_+(k)$ by $F_\#$. There is a one-to-one correspondence between pairs (M, T) over B and homotopy classes of stratified maps $[B; H_+(k)]$ of local orbit spaces. (Homotopies are through equivariant stratified maps.)

Let M be a k-axial $G(n)$ manifold with $G(n) = O(n)$ or $SO(n)$. Let $M_{(i)} = M^{G(n-i)}$. We have $O(i)$ or $S(O(i) \times O(1))$ acting on $M_{(i)}$ according to whether $G(n) = O(n)$ or $SO(n)$. Let $M'_{(i)}$ be the union of the closed top two strata of $M_{(i)}$, i.e. $M'_{(i)}$ is obtained from $M_{(i)}$ by removing an open equivariant tubular neighborhood of the jth stratum of $M_{(i)}$ for $j \leqslant i-3$. Then $(M'_{(i)})_i \to B_i$ is just the principal $O(i)$ (resp. $S(O(i) \times O(1))$ bundle associated to the ith orbit bundle of M, and $(M'_i)_{i-1} \to B_{i-1}$ is the $(i-1)$st orbit bundle of $M'_{(i)}$ with fibre $O(i)/O(1)$ (resp. $SO(i)$). Now $O(i)$ (resp. $S(O(i) \times O(1))$) acts on these fibres by left multiplication. Let $D_i(M) = M'_{(i)}/SO(i)$. Then

$$D_i(M) = (M'_{(i)})_i/SO(i) \cup (M'_{(i)})_{i-1}/SO(i). \tag{29}$$

Now $SO(i)$ acts transitively on $O(i)/O(1)$ (resp. $SO(i)$), and $O(i)/SO(i)$ (resp. $S(O(i) \times O(1))/SO(i)$) is isomorphic to $O(1)$ so that $D_i(M)$ is a double branched cover of $B_i \cup B_{i-1}$ branched along B_{i-1}.

THEOREM 1.2 (THE HOMOLOGY ISOMORPHISM THEOREM; DAVIS [7, pp. 67, 83]). *Let F: $M \to N$ be an equivariant stratified map of k-axial $G(n)$ manifolds. Then F is an integral homology equivalence if and only if the induced map $F_\#$: $B(M) \to B(N)$ induces integral homology equivalences $(F_\#)_i$: $B_i(M) \to B_i(N)$ if $G(n) = U(n)$, or* *integral homology equivalences* $(F_{\#})_i$: $D_i(M) \to D_i(N)$ *for* $(n-i)$ *even if* $G(n) = O(n)$. *If* $G(n) = O(n)$, *then* F *is a* $\mathbf{Z}_2$ *homology equivalence if and only if* $(F_{\#})_i$: $B_i(M) \to B_i(N)$ *is.*

THEOREM 1.3. *Let* F: $M \to N$ *be a stratified map of k-axial* $SG(n)$ *manifolds with* $k \leqslant n-1$. *Then the conclusions of Theorem* 1.2 *apply to* F.

PROOF. With $k \leqslant n-1$, k-axial local $SG(n)$ orbit spaces are the same as in the $G(n)$ case and the proof of Theorem 1.2 carries over with the obvious substitutions.

PROPOSITION 1.1 (DAVIS [7, pp. 67–84]). *Let* Σ *be a homotopy sphere with a k-axial* $G(n)$ *action with orbit space* $B(\Sigma)$. *Then the top stratum* $B_k(\Sigma)$ *of* $B(\Sigma)$ *is contractible.*

PROPOSITION 1.2. *Let* Σ *be an* $(n-1)$*-axial* $SG(n)$ *homotopy sphere with the orbit space* $B(\Sigma)$. *Then* $B_k(\Sigma)$ *is contractible.*

PROOF. Regard $O(n-1) \subset SO(n)$ via $O(n-1) = S(O(n-1) \times O(1))$. Then $O(n-1)$ acts $(n-1)$-axially on Σ. By Proposition 1.1, $\Sigma/O(n-1)$ is acyclic, hence $\Sigma_k/O(n-1)$ is acyclic since it is the interior of $\Sigma/O(n-1)$ which is a manifold with boundary.

The map $\Sigma_k/O(n-1) \to B_k(\Sigma)$ is a fibre bundle projection. Its fibre is the orbit space of the $O(n-1)$ action on the Stiefel manifold $SO(n) \cong O(n)/O(1)$ of $(n-1)$-frames in $\mathbf{R}^n$. By [7, Lemma 10, p. 78] this is topologically a disc of dimension $(n-1)$, hence $B_k(\Sigma)$ is acyclic. To prove $\pi_1 B_k = 0$, let f: $I \to B_k$ be a loop. By path lifting (Bredon [3, p. 90]) f lifts to a path f: $I \to \pi^{-1}(B_k)$. Since $\pi\tilde{f}(0) = \pi\tilde{f}(1)$ and the fibre $\pi^{-1}f(0)$ is connected we can suppose f lifts to a loop $\pi^{-1}B_k$. Let F: $I \times I \to \Sigma$ be a null homotopy. Since $\Sigma - \Sigma_k$ has codimension $\geqslant 3$ we can suppose $\operatorname{Im} F \subset \Sigma_k$. Then $\pi \circ F$ is a nullhomotopy of f so $\pi_1 B_k = 0$. This completes the proof.

PROPOSITION 1.3. *Let* M *be a k-axial* $G(n)$ *manifold, where* $G(n) = SO(n)$, $O(n)$, $SU(n)$ *or* $U(n)$. *Let* π: $M \to B$ *be the orbit map for* M *and suppose* $\pi_1 B = 0$. *Then* $\pi_1 M = 0$.

PROOF. If $G(n) = O(n)$ or $U(n)$, see Davis [7]. Let f: $I \to M$ be a loop in M. Since the codimension of $M - M_k$ in M is greater than or equal to two, f is homotopic to a loop in M_k. Since B is a manifold with boundary B and interior B_k, $\pi_1 B_k = 0$. Hence $\pi \circ f$ is nullhomotopic. Since $M_k \to B_k$ is a fibre bundle, a nullhomotopy of $\pi \circ f$ lifts to a homotopy from f to g with $\pi \circ g(t) = b\ \forall t \in I$, for some fixed $b \in B_k$. Now choose a path from b to b' with $b' \in B_{k-1}$. Lifts of this path give a homotopy from g to g' with $\pi \circ g'(t) = b'$, $b' \in B_{k-1}$. But $\pi^{-1}(b') \cong G(n)/G(2)$, which is simply connected. Therefore g' is homotopic to a constant map. This completes the proof.

Let S be the unit sphere in $M(n,k) \times \mathbf{R}^m$. For local $G(n)$ orbit spaces B, B' let $[B; B']$ denote equivalence classes of stratified maps $B \to B'$ of local orbit spaces, where F_1, F_2 are defined to be equivalent if there exists H: $B \times I \to B'$ such that $H|_{B\times\{t\}}$ is a stratified map of local orbit spaces, $H|_{B\times\{0\}} = F_1$, and $H|_{B\times\{1\}} = F_2$.

THEOREM 1.4 (DAVIS [7, CHAPTER 5]). *Let B be the orbit space of a k-axial $G(n)$ homotopy sphere Σ, $\dim F(\Sigma; G(n)) = m - 1$, with $G(n) = O(n)$ or $U(n)$, and $k \leqslant n$. Then there exists an equivariant stratified map F:*

$$\begin{array}{ccc} \Sigma & \xrightarrow{F} & S \\ \downarrow & & \downarrow \pi \\ B & \xrightarrow{F_{\#}} & B(S) \end{array} \tag{30}$$

F is unique up to isotopy through such maps, and k-axial $G(n)$ manifolds M over B are in one-to-one correspondence with $[B; B(S)]$, M corresponding to the pullback of π via $F_{\#}$. M is a homotopy sphere if and only if $F_{\#}$ induces integral homology isomorphisms $D_i(M) \to D_i(S)$ of the double branched covers for $i \equiv n \pmod 2$ if $G(n) = O(n)$ or $F_{\#}$ induces integral homology isomorphisms $B_i \to B_i(S)$ if $G(n) = U(n)$.

THEOREM 1.5. *Let B, Σ, S, $G(n)$ and k be as in the previous theorem. Suppose that $F(\Sigma, G(n)) = F(S, G(n)) = S^{m-1} \neq \varnothing$ and that for $i > 0$ the ith twist invariant of Σ is a homotopy equivalence. Suppose that $\dim B_i(S) \neq 3$ for $i > 0$. Then Σ is $G(n)$ equivariantly diffeomorphic to S if and only if B is diffeomorphic to $B(S)$ as a stratified space.*

PROOF. An inverse to the classifying map F of the previous theorem can be constructed to be the identity on the fixed point set, then extended stratum by stratum as in Davis [**8**] using the fact that the ith twist invariants for $i > 0$ are homotopy equivalences. This shows that $F_{\#}$ induces homotopy equivalences on each stratum. Suppose that B is diffeomorphic to $B(S)$, without loss of generality suppose $B = B(S)$. In the proof of Theorem 1.4, we may choose $(F_{\#})_0$ to be the identity. Since $B(S)$ is acyclic the normal invariant of $F_{\#}$ vanishes (as in [**11, 13**]). We construct a normal bordism $G_{\#}$ from $F_{\#}$ to the identity. Following Browder and Quinn [**5**] the normal bordism may be constructed stratum by stratum. In particular, we choose the normal bordism from $(F_{\#})_0$ (the identity) to the identity to be the identity on $B_0(S) \times I$.

If $G(n) = U(n)$ we now apply the surgery of [**11**] to the normal bordism. The only surgery obstruction is $\sigma(G_0)_{\#}$ which vanishes by construction. The surgery may be done only on the first and higher strata because of our choice of $(G_0)_{\#}$. (Thus if $\dim B_i(S) \neq 3$ for $i > 0$ we avoid 4-dimensional surgery problems.)

Now as in [**11**] the result follows.

If $G(n) = O(n)$ then $F_{\#}$ satisfies stronger conditions than usual (cf. Theorem 1.4). The proof then consists of strengthening (to $\mathbf{Z}$ and $\mathbf{Z}[\mathbf{Z}/2]$ coefficients) the surgery arguments of [**13**] in our situation, showing that the surgery obstructions not a priori vanishing i.e. $\sigma(G_1)_{\#}$, $\sigma(G_0)_{\#}$ actually do vanish. Then applying a stratified h-cobordism theorem using the fact that $\mathrm{Wh}(\pi_1 B_i) = 0$) we obtain the result. More details will appear in [**2**].

The converse is trivial.

2. The obstruction theory for extending $(n-1)$-axial $SG(n)$ actions to $G(n)$ actions. Let M be a k-axial $SG(n)$ manifold. We first observe that the data, N_i, P_i, f^{ij}, etc., of the normal system (see Definition 1.1) of M is the same as that of a k-axial $G(n)$ manifold up to the $(k-2)$nd stratum (resp. kth stratum) if $k = n-1$ (resp. $k \leqslant n-2$). Assembling the N_i for $i \leqslant k-2$ (resp. $i \leqslant k$) using the data of the normal system of M gives a unique extension to a k-axial $G(n)$ action on $\operatorname{cl}(M - N_k - N_{k-1})$ (resp. M) by regarding the data as part of the normal system of a $G(n)$ manifold. Thus k-axial $SG(n)$ manifolds are in one-to-one correspondence with k-axial $G(n)$ manifolds if $k \leqslant n-2$.

For the rest of this paper let $k = n-1$. This is the least k for which k-axial $SG(n)$ manifolds differ from k-axial $G(n)$ manifolds as exemplified by the biaxial $SO(3)$ action of Gromoll and Meyer [**14**] on the exotic seven sphere $\Sigma^7_{(2,-1)}$ of Milnor [**17**], which cannot pull back from the linear model. This is also the least k for which k-axial $SG(n)$ manifolds have trivial principal isotopy subgroup and hence lie outside the scope of Davis, Hsiang and Hsiang [**12**].

Let M be an $(n-1)$-axial $SG(n)$ manifold.

LEMMA 2.1. *Equivariant stratified isotopy classes of $(n-1)$-axial $G(n)$ actions which extend the original $SG(n)$ action on $N_i \subset M$ are in one-to-one correspondence with homotopy classes of reductions of the structure group of P_i from S_i to S'_i, where S'_i denotes the ith structure group of an $(n-1)$-axial $G(n)$ manifold. Similar statements hold with N_i replaced by ∂N_i and P_i replaced by ∂P_i.*

PROOF. The correspondence is given as follows: a reduction

$$P'_i \to P_i \tag{1}$$

of the structure group of P_i determines an $SG(n)$ equivariant diffeomorphism

$$\begin{aligned} N'_i &= \{G(n) \times_{G(n-i)} D(n-i, k-i)\} \times_{S'_i} P'_i \\ &\xrightarrow{\cong} \{SG(n) \times_{SG(n-i)} D(n-i, k-i)\} \times_{S_i} P_i = N_i, \end{aligned} \tag{2}$$

where N'_i is a $G(n)$ manifold. This determines an $(n-1)$-axial $G(n)$ action on N_i. Conversely to such an action there is a diffeomorphism $N'_i \cong N_i$ as above which induces a reduction $P'_i \to P_i$.

A homotopy of reductions of the structure group clearly gives rise to an equivariant stratified isotopy $N'_i \times I \to N_i$ and vice versa. This completes the proof.

Henceforth we regard two extensions of an $SG(n)$ action to $G(n)$ which differ by an $SG(n)$-equivariant stratified isotopy, as the same.

2.1. *Extensions over the second to top stratum.* Since $\partial N_{k-1}(M)$ is contained in $\operatorname{cl}(M - N_k(M) - N_{k-1}(M))$ we have an extension of the action to $G(n)$ over $\partial N_{k-1}(M)$ and hence have a reduction of the structure group of $\partial P_{k-1}(M)$ from $S_{k-1} = S(G(n-2) \times G(2))$ to $S'_{k-1} = G(n-2) \times G(1) \cong S(G(n-2) \times G(1) \times G(1))$. Extensions of the $G(n)$ action to $\operatorname{cl}(M - N_k(M))$ therefore correspond to extensions of the reduction of $\partial P_{k-1}(M)$ to a reduction of $P_{k-1}(M)$. Since reductions of the structure groups of a principal H bundle to $K \subset H$ correspond to sections of the associated H/K bundle, we must consider the relative section problem for the associated S_{k-1}/S'_{k-1} bundle to $P_{k-1}(M)$.

LEMMA 2.2. *If $SG(n) = SO(n)$ and $G(n) = O(n)$, then*:

(i)

(3) $$S_{k-1}/S'_{k-1} = S(O(n-2) \times O(2))/S(O(n-2) \times O(1) \times O(1))$$

which is homotopy equivalent to S^1.

(ii) *There is a single obstruction in $H^2(B_{k-1}, \partial B_{k-1}; \underset{\sim}{\mathbf{Z}})$ to the existence of a section of $P_{k-1}(M)/S'_{k-1}$, and such extensions are in one-to-one correspondence with $H^1(B_{k-1}, \partial B_{k-1}; \underset{\sim}{\mathbf{Z}})$, where $\underset{\sim}{\mathbf{Z}}$ denotes integer coefficients twisted by the action of $\pi_1 B_{k-1}(M)$ on $\pi_1 S^1 = \mathbf{Z}$ which is determined by a map $\pi_1 B_{k-1}(M) \to S_{k-1}$ arising from the principal bundle $P_{k-1} \to B_{k-1}$.*

PROOF. Immediate from (11) and (12) of §1 and classical obstruction theory.

LEMMA 2.3. *If $SG(n) = SU(n)$ and $G(n) = U(n)$, then*:

(i)

(4) $$S_{k-1}/S'_{k-1} = S(U(n-2) \times U(2))/S(U(n-2) \times U(1) \times U(1))$$

which is homotopy equivalent to S^2.

(ii) *There are obstructions in*

(5) $$H^{i+1}\big(B_{k-1}(M), \partial B_{k-1}(M); \pi_i S^2\big)$$

to existence and obstructions in

(6) $$H^{i}\big(B_{k-1}(M), \partial B_{k-1}(M); \pi_i S^2\big)$$

to uniqueness for the problem of extending sections of $P_{k-1}(M)/S'_{k-1}$ from $\partial B_{k-1}(M)$ to $B_{k-1}(M)$.

(iii) *The system of coefficients $\pi_i S^2$ is untwisted for all i.*

PROOF. (i) This follows from (11) and (12) of §1.

(ii) This is immediate from (i) and classical obstruction theory.

(iii) The action of $\pi_1 B_{k-1}(M)$ on $\pi_i S^2$ is given from a homomorphism $\pi_1 B_{k-1}(M) \to S(U(k-1) \times U(2))$. $S(U(k-1) \times U(2))$ is connected and hence acts trivially on $\pi_i S^2$. The lemma follows.

2.2. *Extensions over the top stratum.*

LEMMA 2.4. *Let M be an $(n-1)$-axial $SG(n)$ homotopy sphere together with an extension of the $SG(n)$ action on M to an $(n-1)$-axial $G(n)$ action on* $\mathrm{cl}(M - N_k(M))$. *Let*

$$r = \dim B(M).$$

Then there is a single obstruction in $\pi_{r-1}(SG(n)/G(n-1))$ to extending the $G(n)$ action to an $(n-1)$-axial $G(n)$ action on M, and such extensions are in one-to-one correspondence with $\pi_r(SG(n)/SG(n-1))$.

PROOF. This follows from Lemma 2.1 if one notes that the base space $B_k(M)$ of $P_k(M)$ is a contractible manifold with boundary and has dimension r. The $G(n)$ action on $\partial N_k(M) \subset \mathrm{cl}(M - N_k(M))$ gives a reduction of the structure group of ∂P_k from $S_k = SG(n)$ to $S'_k = G(n-1)$. There is a single obstruction in $\pi_{r-1}(S_k/S'_k)$ to extending this reduction to a reduction of $P_k \cong B_k(M) \times SG(n)$ and such extensions are in one-to-one correspondence with $\pi_r(S_k/S'_k)$. This completes the proof.

3. The second to top stratum.

3.1. *$SO(n)$ actions on homotopy spheres with fixed points.* In this section we show that for $SO(n)$ actions with fixed points the obstruction groups $H^i(B_{k-1}, \partial B_{k-1}; \underset{\sim}{\mathbf{Z}})$, $i = 1, 2$, of subsection 2.1 vanish except when $k = 2$ and the fixed point set has dimension zero (biaxial $SO(3)$ actions on S^6), in which case the above group is isomorphic to $\mathbf{Z}_2$ and the obstruction can be realized by a direct construction.

3.1.1. *The twisted coefficients.* The twisted integer coefficients $\underset{\sim}{\mathbf{Z}}$ for the problem of finding a section of

$$S^1 \sim S(O(k-1) \times O(2))/O(k-1) \times O(1) \to P_{k-1}/O(k-1) \times O(1) \to B_{k-1}$$

correspond to a double cover of B_{k-1} via the homomorphism

$$\begin{aligned} \pi_1 B_{k-1} &\to S(O(k-1) \times O(2)) \\ &\to S(O(k-1) \times O(2))/SO(k-1) \times SO(2) \cong \mathbf{Z}_2, \end{aligned}$$

where the first map is determined by the bundle P_{k-1} and the second map is the quotient by the identity component of $S(O(k-1) \times O(2))$. The double cover is

$$P_{k-1}/SO(k-1) \times SO(2).$$

LEMMA 3.1.

$$\tilde{B}_{k-1} = P_{k-1}/SO(k-1) \times SO(2), \tag{1}$$

where $\tilde{B}_{k-1} = (M'_{(k-1)})_{k-1}/SO(k-1)$ is the double cover of B_{k-1} occurring in the Davis double branched covers (see §1, (29)).

PROOF.

$$\tilde{B}_{k-1} = (M_{k-1})^{SO(2)}/SO(k-1), \tag{2}$$

$$N_{k-1} = \{SO(n) \times_{SO(2)} M(2,1)\} \times_{S(O(k-1)\times O(2))} P_{k-1}, \tag{3}$$

$$M_{k-1} = \{SO(n) \times_{SO(2)} \{0\}\} \times_{S(O(k-1)\times O(2))} P_{k-1}, \tag{4}$$

so

$$\begin{aligned} (M_{k-1})^{SO(2)} &= N_{SO(n)}(SO(2))/SO(2) \times_{S(O(k-1)\times O(2))} P_{k-1} \\ &= S(O(k-1) \times O(2))/SO(2) \times_{S(O(k-1)\times O(2))} P_{k-1} \\ &= P_{k-1}/SO(2), \end{aligned} \tag{5}$$

and

$$\tilde{B}_{k-1} = P_{k-1}/SO(k-1) \times SO(2), \tag{6}$$

as required.

3.1.2. *Some orientable fibrations.*

LEMMA 3.2. *The fibrations*

$$\partial_i B_j \to B_j, \tag{7}$$

$$\partial_i \partial_l B_j \to \partial_l B_j \tag{8}$$

are orientable for all i, j, l with $0 \leqslant j \leqslant k, j < i \leqslant k, j \leqslant l \leqslant k$ and $l \neq i$.

PROOF. The normal cone bundle

$$H_+(k-j) \to C_j \to B_j \tag{9}$$

has structure group $O(k-j)$ which acts on $H_+(k-j)$ by conjugation.

$\partial_i B_j$ is the associated bundle with fibre

$$H_+(k-j)_{(i-j)} \cong H_+(i-j) \times_{O(i-j)\times O(k-j-(i-j))} O(k-j) \tag{10}$$

with the identification being

$$[x, g] \leftrightarrow g^{-1}\begin{pmatrix} x & \\ & 0 \end{pmatrix} g. \tag{11}$$

So we can regard $O(k-j)$ as acting on $O(i-j) \times O(k-i) \backslash O(k-j)$ by

$$h \cdot O(i-j) \times O(k-i) g = O(i-j) \times O(k-i) g h^{-1}, \tag{12}$$

where $h, g \in O(k-j)$. If $h \in SO(k-j)$, then h acts trivially on the homology of $O(i-j) \times O(k-i) \backslash O(k-j)$ since there is a path from h to the identity. If $h \in O(i-j)$, then in (12) gh^{-1} may be replaced by hgh^{-1} which implies h acts trivially on the homology of $O(i-j) \times O(k-i) \backslash O(k-j)$. In general, h is a product of $h_1 \in \in SO(k-j)$ and $h_2 \in O(i-j)$, so h acts trivially on homology always. This proves the lemma.

LEMMA 3.3. *Let $f: M \to M'$ be an equivariant stratified map of $(n-1)$-axial $SO(n)$ manifolds, and suppose that*

$$f_*: H_*(M; \mathbf{Z}) \overset{\cong}{\to} H_*(M'; \mathbf{Z}) \tag{13}$$

is an isomorphism in all dimensions. Then

$$\begin{aligned} f^*: & H^*(B_{k-1}(M'), \partial B_{k-1}(M'); \underset{\sim}{\mathbf{Z}}) \\ & \overset{\cong}{\to} H^*(B_{k-1}(M), \partial B_{k-1}(M); \underset{\sim}{\mathbf{Z}}). \end{aligned} \tag{14}$$

PROOF. Our starting point shall be one of the computations (Davis [7, Appendix]) involved in the proof of the Double Branched Cover Homology Isomorphism Theorem for $O(n)$ actions, which generalizes to $SO(n)$ actions without change. We also use the Double Branched Cover Homology Isomorphism Theorem for multiaxial $SO(n)$ actions (Theorem 1.3).

Let ν be the normal disc bundle of $B_{k-2}(M)$ in $B_{k-1}(M)$, and let $\Sigma\nu$ be the unit sphere bundle. By the hypothesis f must induce homology isomorphisms of the double branched cover $D_i(M)$ for all $i \leqslant n+1$ with $n-i$ even. Davis calculates

$$H_i\left(f|_{\tilde{B}_{k-1}(M)}\right) \cong H_{i+1}\left(f|_{D_{k-1}(M)}, f|_{\tilde{B}_{k-1}}\right) \cong H_{i+1}(f|_\nu, f|_{\Sigma\nu}). \tag{15}$$

The first isomorphism is the boundary operator in the exact sequence of the pair $(f|_{D_{k-1}(M)}, f|_{\tilde{B}_{k-1}(M)})$ and holds since

$$H_*\left(f|_{D_{k-1}(M)}; \mathbf{Z}\right) = 0 \tag{16}$$

by the Double Branched Cover Homology Isomorphism Theorems 1.2 and 1.3. The second isomorphism of (15) is excision.

By the universal coefficient theorems

$$H^*\left(f|_{D_{k-1}(M)};\mathbf{Z}\right)=0 \tag{17}$$

and the sequence (15) dualizes viz:

$$H^i\left(f|_{\tilde{B}_{k-1}(M)}\right)\overset{\delta}{\cong}H^{i+1}\left(f|_{D_{k-1}(M)},f|_{\tilde{B}_{k-1}(M)}\right)\cong H^{i+1}(f|_\nu,f|_{\Sigma\nu}). \tag{18}$$

Now $\mathbf{Z}_2$ acts by the antipodal map on the fibre of ν, which has dimension 2, which is even, so $\mathbf{Z}_2$ acts trivially on $H^*(f|_{\Sigma\nu})$ and on $H^*(f|_\nu)$, and hence on $H^*(f|_\nu, f|_{\Sigma\nu})$.

For a $\mathbf{Z}_2$ equivariant map $f\colon X\to Y$ between spaces with $\mathbf{Z}_2$ action let $H_i(f|_X)^-$ denote the subgroup of $H_i(f|_X)$ on which the generator of $\mathbf{Z}_2$ acts by multiplication by -1. We have shown that

$$H^i(f|_{\Sigma\nu})^-=H^i\left(f|_{\tilde{B}_{k-1}}\right)^-=H^{i+1}(f|_\nu,f|_{\Sigma\nu})^-=0 \tag{19}$$

but

$$H^i\left(f|_{B_{k-1}};\underset{\sim}{\mathbf{Z}}\right)\cong H^i\left(f|_{\tilde{B}_{k-1}}\right)^-=0. \tag{20}$$

Furthermore, since $\Sigma\nu=\partial_{k-2}\tilde{B}_{k-1}$, we obtain

$$H^i\left(f|_{\partial_{k-2}B_{k-1}},\underset{\sim}{\mathbf{Z}}\right)\cong H^i\left(f|_{\Sigma\nu}\right)^-=0. \tag{21}$$

The same argument with k replaced by $j+1$ shows that, for $n-j$ even,

$$H^*\left(f|_{B_j};\underset{\sim}{\mathbf{Z}}\right)=0 \tag{22}$$

and

$$H^*\left(f|_{\partial_{j-1}B_j};\underset{\sim}{\mathbf{Z}}\right)=0, \tag{23}$$

where the twisted coefficients correspond to the double covers $\tilde{B}_j\to B_j$.

SUBLEMMA. *Let $n-j$ be an even integer. Under the identifications*

$$\partial_j B_{k-1}=\partial_{k-1}B_j, \tag{24}$$

the two double covers

$$\begin{array}{ccc}\partial_j\tilde{B}_{k-1} & & \partial_{k-1}\tilde{B}_j\\ \downarrow & \textit{and} & \downarrow\\ \partial_j B_{k-1} & & \partial_{k-1}B_j\end{array} \tag{25}$$

are the same, where $\partial_{k-1}B_j=\partial_{k-1}[(N_j)^{SO(n-j)}/SO(j)]$.

PROOF. The commutative diagram

$$\begin{array}{ccc}SO(n)/SO(n-j) & \hookrightarrow & SO(n)/SO(2)\\ \downarrow & & \downarrow\\ \partial_{k-1}N_j & \hookrightarrow & M_{k-1}\\ \downarrow & & \downarrow\\ \partial_{k-1}B_j & \hookrightarrow & B_{k-1}\end{array} \tag{26}$$

induces a diagram

$$\begin{array}{ccc} O(j) & \hookrightarrow & O(n-2) \\ \downarrow & & \downarrow \\ \partial_{k-1}(N_j)^{SO(n-j)} & \hookrightarrow & (M_{k-1})^{SO(2)} \\ \downarrow & & \downarrow \\ \partial_{k-1}B_j & \hookrightarrow & B_{k-1} \end{array} \tag{27}$$

which induces:

$$\begin{array}{ccc} \mathbf{Z}_2 = O(j)/SO(j) & = & O(n-2)/SO(n-2) \\ \downarrow & & \downarrow \\ \partial_{k-1}\tilde{B}_j & \hookrightarrow & (M_{k-1})^{SO(2)}/SO(n-2) = \tilde{B}_{k-1} \\ \downarrow & & \downarrow \\ \partial_{k-1}B_j & \hookrightarrow & B_{k-1} \end{array} \tag{28}$$

This shows that $\partial_{k-1}\tilde{B}_j$ is the part of $\tilde{B}_{k-1}$ over $\partial_{k-1}B_j = \partial_j B_{k-1}$, which is precisely $\partial_j \tilde{B}_{k-1}$. This proves the Sublemma.

Let $n-j$ be even in the remainder of the proof of this lemma. By Lemma 3.2, f induces a bundle map of *orientable* fibrations:

$$\begin{array}{ccc} \partial_{k-1}B_j(M) & \to & \partial_{k-1}B_j(M') \\ \downarrow & & \downarrow \\ B_j(M) & \to & B_j(M') \end{array} \tag{29}$$

Since we know already that $H_*(f|_{B_j(M)}; \underset{\sim}{\mathbf{Z}}) = 0$, the comparison theorem for spectral sequences applies to show that

$$H^*\left(f|_{\partial_{k-1}B_j(M)}; \underset{\sim}{\mathbf{Z}}\right) = 0. \tag{30}$$

Since $\partial_{k-1}\tilde{B}_j(M) = \partial_j\tilde{B}_{k-1}(M)$ by the Sublemma, it follows from (30) that

$$H^*\left(f|_{\partial_j B_{k-1}(M)}; \underset{\sim}{\mathbf{Z}}\right) = 0. \tag{31}$$

The map of orientable fibrations

$$\begin{array}{ccc} \partial_j \partial_{j-1}B_{k-1}(M) & \to & \partial_j \partial_{j-1}B_{k-1}(M') \\ \downarrow & & \downarrow \\ \partial_j B_{k-1}(M) & \to & \partial_j B_{k-1}(M') \end{array} \tag{32}$$

yields

$$H^*\left(f|_{\partial_j\partial_{j-1}B_{k-1}(M)}; \underset{\sim}{\mathbf{Z}}\right) = 0. \tag{33}$$

Then the map of orientable fibrations

$$\begin{array}{ccc} \partial_j \partial_{j-1}B_{k-1}(M) & \to & \partial_j \partial_{j-1}B_{k-1}(M') \\ \downarrow & & \downarrow \\ \partial_{j-1} B_{k-1}(M) & \to & \partial_{j-1} B_{k-1}(M') \end{array} \tag{34}$$

yields

$$(35) \qquad H^*\left(f|_{\partial_{l-1}B_{k-1}(M)}; \underset{\sim}{\mathbf{Z}}\right) = 0.$$

Finally, the map of orientable fibrations

$$(36) \qquad \begin{array}{ccc} \partial_l\partial_j B_{k-1}(M) & \to & \partial_l\partial_j B_{k-1}(M') \\ \downarrow & & \downarrow \\ \partial_j B_{k-1}(M) & \to & \partial_j B_{k-1}(M') \end{array}$$

yields

$$(37) \qquad H^*\left(f|_{\partial_l\partial_j B_{k-1}(M)}; \underset{\sim}{\mathbf{Z}}\right) = 0.$$

Now

$$(38) \qquad \partial B_{k-1}(M) = \bigcup_{j<k-1} \partial_j B_{k-1}(M)$$

and

$$(39) \qquad \partial_j B_{k-1}(M) \cap \partial_l B_{k-1}(M) = \partial_j\partial_l B_{k-1}(M).$$

We have shown that f induces an isomorphism on each piece and each intersection, so by induction and Mayer-Vietoris sequences we obtain

$$(40) \qquad H^*\left(f|_{\partial B_{k-1}}; \underset{\sim}{\mathbf{Z}}\right) = 0.$$

Since

$$(41) \qquad H^*\left(f|_{B_{k-1}}; \underset{\sim}{\mathbf{Z}}\right) = 0$$

the exact sequence of the pair yields

$$(42) \qquad H^*\left(f|_{B_{k-1}}, f|_{\partial B_{k-1}}; \underset{\sim}{\mathbf{Z}}\right) = 0.$$

This completes the proof of the lemma. We now apply Lemma 3.3 to compute $H^*(B_{k-1}(\Sigma), \partial B_{k-1}(\Sigma); \underset{\sim}{\mathbf{Z}})$ for an $(n-1)$-axial $SO(n)$ homology sphere Σ with fixed points. Let p, q be distinct fixed points of Σ. Let N_p and N_q be invariant neighborhoods of p and q respectively. Then

$$(43) \qquad \Sigma = \mathrm{cl}(\Sigma - N_p) \cup N_p.$$

The inclusion $N_q \hookrightarrow \mathrm{cl}(\Sigma - N_p)$ is a homology isomorphism hence

$$(44) \qquad \begin{aligned} &H^*\left(B_{k-1}(\mathrm{cl}(\Sigma - N_p)), \partial B_{k-1}(\mathrm{cl}(\Sigma - N_p)); \underset{\sim}{\mathbf{Z}}\right) \\ &\quad\cong H^*\left(B_{k-1}(N_q), \partial B_{k-1}(N_q); \underset{\sim}{\mathbf{Z}}\right) \cong 0. \end{aligned}$$

The last isomorphism in (44) holds since N_q is equivariantly diffeomorphic to the linear model $M(n,k) \times \mathbf{R}^m$ so that $\partial B_{k-1}(N_q) \hookrightarrow B_{k-1}(N_q)$ is a homotopy equivalence.

Now we have an excisive couple of pairs

$$(45)\quad \begin{aligned}\left(B_{k-1}(\mathrm{cl}(\Sigma - N_p)), \partial B_{k-1}(\mathrm{cl}(\Sigma - N_p))\right) \cup \left(B_{k-1}(N_p), \partial B_{k-1}(N_p)\right)\\ = \left(B_{k-1}(\Sigma), \partial B_{k-1}(\Sigma)\right).\end{aligned}$$

Further,

$$(46)\quad B_{k-1}(\mathrm{cl}(\Sigma - N_p)) \cap B_{k-1}(N_p) = B_{k-1}(\partial N_p).$$

So Mayer-Vietoris sequences applied to the excisive couple yield

$$(47)\quad H^*(B_{k-1}(\Sigma), \partial B_{k-1}(\Sigma); \underset{\sim}{\mathbf{Z}}) \cong H^{*-1}(B_{k-1}(\partial N_p), \partial B_{k-1}(\partial N_p); \underset{\sim}{\mathbf{Z}}).$$

Now, if the dimension of the fixed point set of Σ is $m - 1 \geqslant 0$, then $\partial N_p \cong S^{n(n-1)+m-2}$, the unit sphere in $M(n, n-1) \times \mathbf{R}^{m-1}$. Let S denote $S^{n(n-1)+m-2}$. It remains to compute

$$(48)\quad H^*(B_{k-1}(S), \partial B_{k-1}(S); \underset{\sim}{\mathbf{Z}}).$$

Now

$$(49)\quad B_{k-1}(S) \cong S(H_+(k-1) \times \mathbf{R}^{m-1}) \times_{O(k-1)\times O(1)} O(k)$$

and

$$(50)\quad \partial B_{k-1}(S) \cong S(\partial H_+(k-1) \times \mathbf{R}^{m-1}) \times_{O(k-1)\times O(1)} O(k).$$

LEMMA 3.4.

$$H^i(B_{k-1}(S), \partial B_{k-1}(S); \underset{\sim}{\mathbf{Z}}) = H^{i-r(k,m)}(\mathbf{R}P^{k-1}; \mathbf{Z}^t),$$

where $r(k, m) = \frac{1}{2}k(k-1) + m - 1$, *and* $H^*(\mathbf{R}P^{k-1}; \mathbf{Z}^t)$ *denotes the cohomology of* $\mathbf{R}P^{k-1}$ *with coefficients twisted by the nontrivial action of* $\pi_1 \mathbf{R}P^{k-1}$ *on* $\mathbf{Z}$.

PROOF. The following are orientable fibrations:

$$(51)\quad S(H_+(k-1) \times \mathbf{R}^m \to \partial B_{k-1}(S) \to \mathbf{R}P^{k-1},$$

$$(52)\quad \begin{aligned}(S(H_+(k-1) \times \mathbf{R}^m), S(\partial H_+(k-1) \times \mathbf{R}^m))\\ \to (B_{k-1}(S), \partial B_{k-1}(S)) \to \mathbf{R}P_{k-1}.\end{aligned}$$

Orientability for (51) follows from the commutative diagram
(53)

$$\begin{array}{ccc} S(\partial H_+(1) \times \mathbf{R}^m) \times_{O(1)\times O(1)} O(2) & \xrightarrow{i} & S(\partial H_+(k-1) \times \mathbf{R}^m) \times_{O(k-1)\times O(1)} O(k) \\ \downarrow p_1 & & \downarrow p_{k-1} \\ \mathbf{R}P^1 & \xrightarrow{i} & \mathbf{R}P^{k-1} \end{array}$$

since $i_\#: \pi_1 \mathbf{R}P^1 \to \pi_1 \mathbf{R}P^{k-1}$ is onto and p_1 is an orientable fibration. Orientability for (52) follows the same way.

Observe that, since the double cover $\tilde{B}_{k-1}(S)$ is connected (see (2)) the coefficients $\underset{\sim}{\mathbf{Z}}$ are nontrivially twisted. The lemma now follows from the fibration (52).

THEOREM 3.1. *Let $n \geqslant 3$ and let Σ be an $(n-1)$-axial $SO(n)$ homotopy sphere with fixed point set of dimension $m - 1 \geqslant 0$. Recall that $B_{k-1}(\Sigma)$ is the second to top stratum of $B(\Sigma)$. The obstruction groups*

$$H^i(B_{k-1}(\Sigma), \partial B_{k-1}(\Sigma); \underset{\sim}{\mathbf{Z}}), \qquad i = 1, 2, \tag{54}$$

are trivial except in the case of biaxial $SO(3)$ actions on S^6 with zero-dimensional fixed point set, in which case

$$H^1(B_{k-1}(\Sigma), \partial B_{k-1}(\Sigma); \underset{\sim}{\mathbf{Z}}) = 0 \quad \text{and} \quad H^2(B_{k-1}(\Sigma), \partial B_{k-1}(\Sigma); \underset{\sim}{\mathbf{Z}}) = \mathbf{Z}_2.$$

The nontrivial element of $\mathbf{Z}_2$ can be realized by a biaxial action of S^6.

PROOF. The calculation of the obstruction groups follows from the lemma above together with the known twisted cohomology of $\mathbf{R}P_{k-1}$. The last sentence follows from a direct computation of $SO(3)$ and $O(3)$ equivariant self-equivalences of $S^5 = S(M(3,2))$ which has only two strata (see Bredon [**3**]). Gluing two copies of $D^6 = D(M(3,2))$ by the appropriate $SO(3)$ equivalence which is not an $O(3)$ equivalence, yields the required example.

3.2. *$SU(n)$ actions on homotopy spheres with fixed points.*

LEMMA 3.5. *Let Σ be an $(n-1)$-axial $SU(n)$ homotopy sphere with fixed point set of dimension $m - 1 \geqslant 0$. Then there is a unique extension of the action on Σ to an $(n-1)$-axial $U(n)$ action on* $\mathrm{cl}(\Sigma - N_q)$ *and on N_q.*

PROOF. Let p, q be distinct fixed points and let N_p and N_q be invariant neighborhoods of p and q in Σ. By §2 we may suppose that we already have the $U(n)$ action on $\mathrm{cl}(\Sigma - N_k(\Sigma) - N_{k-1}(\Sigma))$.

Consider the inclusion

$$i\colon N_p \hookrightarrow \mathrm{cl}(\Sigma - N_q), \tag{55}$$

which is an equivariant stratified integral homology equivalence. By the Homology Isomorphism Theorem (Theorem 1.3)

$$\begin{aligned} i^*\colon & H^*\big(B_{k-1}\big(\mathrm{cl}(\Sigma - N_q)\big), \partial B_{k-1}\big(\mathrm{cl}(\Sigma - N_q)\big); \mathbf{Z}\big) \\ & \xrightarrow{\cong} H^*\big(B_{k-1}(N_p), \partial B_{k-1}(N_p); \mathbf{Z}\big) = 0. \end{aligned} \tag{56}$$

Hence the obstruction theory of §2 applies to show that there is a unique extension up to the second to top strata of the $SU(n)$ action to $U(n)$ on $\mathrm{cl}(\Sigma - N_q)$ and on N_q. A similar argument with the top stratum shows that there is a unique extension to $U(n)$ over $\Sigma - N_q$ and N_q.

Let S be the unit sphere in $M(n, n-1) \times \mathbf{R}^{m-1}$. Let Θ_0, Θ_1 denote the structures on S induced by the identifications $S \cong \partial N_q$ and $S \cong \partial(\Sigma - N_q)$, respectively. Θ_i, $i = 0, 1$, consists of the standard $SU(n)$ action on S together with an extension to $U(n)$. To avoid 4-dimensional surgery problems assume $m - 2 = \dim F(S, SU(n)) \neq 4$, and if $0 \neq m - 1 \leqslant 4$ assume $n \geqslant 4$.

LEMMA 3.6. *Let (S, Θ_i), $i = 0, 1$, be as above. Then (S, Θ_1) is $U(n)$ equivariantly diffeomorphic to (S, Θ_0).*

PROOF. $U(n)$ actions with standard fixed point set of dimension greater than four and with simply connected strata are classified (Davis and Hsiang [**11**]) by the index or Kervaire invariant of their fixed point set, which is zero for (S, Θ_i), $i = 0, 1$.

Let s_i be the sections of $P_{k-1}(S)/U(k-1) \times U(1)$ corresponding to the $U(n)$ actions Θ_i on $(S - N_k(S))$. Note that $\partial s_i = s_i|_{\partial B_{k-1}(s)}$ is standard. Let $c(s_i)$ denote the element of $\pi_{r-1}(S(U(k-2) \times U(2))/U(k-1) \times U(1))$ obtained by collapsing to the top cell of B_{k-1}. Let

(57) $r\colon \pi_{r-1}(S(U(k-2) \times U(2))/U(k-1) \times U(1)) \to \pi_{r-1}(SU(n)/U(k))$

be the natural map. We can view the homotopy class of s_1 (rel ∂B_{k-1}) as the obstruction to extending the $U(n)$ action over $(\Sigma - N_k(\Sigma))$.

LEMMA 3.7. *Let $\bar{s}$ be any reduction of $P_{k-1}(S)$ from $S(U(k-2) \times U(2))$ to $U(k-1) \times U(1)$ with $\partial \bar{s} = \partial s_0$. Then $\bar{s}$ can be realized as the obstruction to extending the $U(n)$ action on $\Sigma - N^k(\Sigma)$ for some $SU(n)$ homotopy sphere Σ if and only if*

(58) $$rc(\bar{s}) = 0 \in \pi_{r-1}(SU(n)/U(k)).$$

PROOF. If $\bar{s}$ can be realized, then the $U(n)$ action on $S - N_k(S)$ (given by $\bar{s}$) extends to a $U(n)$ action (S, Θ_1). Thus the obstruction $rc(\bar{s})$ for this is zero.

Conversely, if $rc(\bar{s}) = 0$ then the $U(n)$ action on $S - N_\kappa(S)$ extends to a $U(n)$ action (S, Θ_1). By Lemma 3.6 there is a $U(n)$ equivariant diffeomorphism

(59) $$\varphi\colon (S, \Theta_1) \to (S, \Theta_0).$$

Let $\Sigma = D \cup_\varphi D$, where D is the unit disc in $M(n, n-1) \times \mathbf{R}^{m-1}$. This realizes $\bar{s}$ as the obstruction to extending the $U(n)$ action on $\Sigma - N_k(\Sigma)$ as required.

Let K denote the homotopy theoretically defined subgroup of homotopy classes of sections s satisfying the hypotheses of Lemma 3.7.

4. The top stratum. In the first part of this section we construct the universal models $S_{\gamma,\sigma}$. The construction involves a diffeomorphism ϕ (which is shown to exist). In the second part we give explicit constructions (involving elements of homotopy groups of $SG(n)$ and $G(n)$) for most ($\partial\gamma \in \operatorname{Im} j_\#$ below) of the elements $S_{\gamma,1}$. These explicit constructions are used in §§6 and 7 below.

LEMMA 4.1. *Let Σ be an $(n-1)$-axial $SG(n)$ manifold with an extension to $G(n)$ over $\Sigma - N_k(\Sigma)$. Let $\gamma \in \pi_{r-1}(SG(n)/G(n-1))$ be the obstruction to extending the $G(n)$ action over Σ.*

(i) *If $SG(n) = SO(n)$ then γ is a well defined invariant of Σ.*

(ii) *If $SG(n) = SU(n)$ then γ has indeterminacy* Im r *where r is the map of* (57). *Moreover by changing the extension to $U(n)$ over $N_{k-1}(\Sigma)$ γ can be modified by adding any desired element of* Im r.

PROOF. (i) This holds since the extension to $O(n)$ over $\Sigma - N_k(\Sigma)$ is unique.

(ii) The normal cone bundle of B_{k-1} is a trivial I ($= [0,1]$) bundle so that $\partial_{k-1}B_k$ may be identified with $B_{k-1} \times \{1\}$. With this identification P_{k-1} is identified with a fixed reduction $\mathfrak{R}$ of the structure group of $\partial_{k-1}P_k$ from $SU(n)$ to $S(U(k-1) \times U(2))$. Since the extension over $\Sigma - N_k(\Sigma) - N_{k-1}(\Sigma)$ is unique the indeterminacy of γ arises from choosing a different section $\bar{s}$ of $P_{k-1}/U(k-1) \times U(1)$ rel ∂B_{k-1}.

This changes the section of $\partial P_k/U(k)$ over $\partial_{k-1}B_k$ rel $\partial\partial_{k-1}B_k$. Now $\partial_{k-1}B_k$ is contained in a disc D^{r-1} in ∂B_k since ∂B_k is a homotopy sphere. Over $D^{r-1} - \partial_{k-1}B_k$ there is no change in the section of $\partial P_k/U(k)$, so that we obtain an element of $\pi_{r-1}SU(n)/U(k)$. The change in ∂P_k consists of the difference of two reductions to $U(k-1) \times U(1)$. Each of these reductions is obtained by first reducing to $S(U(k-1) \times U(2))$ by the reduction $\mathfrak{R}$ then reducing further to $U(k-1) \times U(1)$. Section 3.2 identifies homotopy classes of sections of $P_{k-1}(\Sigma)/U(k-1) \times U(1)$ with π_1 of sections of $P_{k-1}(S)/U(k-1) \times U(1)$. Thus they form a group. Hence the change in $\partial_{k-1}P_k$ is an element of

(1) $$\operatorname{Im}\{[\partial_{k-1}B_k, S(U(k-1) \times U(2))/U(k-1) \times U(1)] \to [\partial_{k-1}B_k, SU(n)/U(k-1) \times U(1)]\}$$

where the map is induced by the inclusion

(2) $$S(U(k-1) \times U(2)) \to SU(n).$$

Thus the change in γ is given by an element of Im r. To modify γ by an element δ of Im r it suffices to change the section of $P_{k-1}/U(k-1) \times U(1)$ by δ on the top cell of B_{k-1}.

THEOREM 4.1. *Let S, D be as in Lemma* 3.7, *and suppose the dimension hypothesis of* §5, (1) *is satisfied by D.*

(i) $\forall \gamma \in \pi_{r-1}(SG(n)/G(n-1))$ *there is an $SG(n)$ equivariant diffeomorphism ϕ: $S \to S$ such that $S_{\gamma,1} = D_1 \cup_\phi D_2$ where $D_1 = D_2 = D$ has a $G(n)$ action over $S_{\gamma,1} - N_k(S_{\gamma,1})$ and obstruction γ to extending to $G(n)$ over $S_{\gamma,1}$.*

PROOF. Let $\gamma \in \pi_{r-1}(SG(n)/G(n-1))$. Let $S - N_k(S)$ have the linear $G(n)$ action. Let (S, Θ_1) be the extension to $G(n)$ corresponding to γ by the correspondence of Lemma 2.4. By Theorem 1.5 a $G(n)$ equivariant diffeomorphism γ: $(S, \Theta_1) \to (S, \Theta_0)$ exists, where (S, Θ_0) is S with the linear action. Now $S_{\gamma,1}$ as defined above has a $G(n)$ action on each D_i, each of which is standard except on the top stratum where they differ by γ on S. Thus γ is the obstruction to extending to $G(n)$ over $S_{\gamma,1}$.

DEFINITION 4.1. If $SG(n) = SU(n)$ then let

(3) $$S_{\gamma,\sigma} = S_{\gamma,1} \# S_{1,\sigma}$$

where $S_{1,\sigma}$ is the space Σ of §3, (58) with $\gamma \in H$ and $\sigma = [\bar{s}] \in K$. If $SG(n) = SO(n)$ then let $S_{\gamma,\sigma}$ be defined by (3) with $S_{1,\sigma} = S$.

THEOREM 4.2. *Let Σ be an $(n-1)$-axial $SG(n)$ homotopy sphere. Suppose that Σ satisfies the hypothesis* (1) *of* §5. *Let (Σ, Θ_0), (Σ, Θ_1) be any two extensions of Σ to $G(n)$. Then (Σ, Θ_0) and (Σ, Θ_1) are $G(n)$ equivariantly diffeomorphic.*

PROOF.

Case 1. $\Sigma = S$: Then the result follows by Theorem 1.5.

Case 2. Arbitrary Σ: We have shown that $\Sigma - N_q$ and N_q have unique extensions to $G(n)$. The extension of Σ to $G(n)$ corresponds to a homotopy of extensions (matching the extensions on $\partial(\Sigma - N_q)$ and ∂N_q). Any two such homotopies differ by a homotopy from the standard extension to itself. This latter homotopy may be used to give an extension of S to $G(n)$ so that

$$(4) \qquad (\Sigma, \Theta_1) = (S, \Theta_1) \# (\Sigma, \Theta_0),$$

and the result follows from Case 1.

Let Σ be an $(n-1)$-axial $SG(n)$ homotopy sphere with fixed points, together with an extension of the action to $G(n)$ over $\mathrm{cl}(\Sigma - N_k(\Sigma))$. Let $\gamma \in \pi_{r-1}(SG(n)/G(n-1))$ be the obstruction to completing the extension to a $G(n)$ action on Σ. We analyze γ in the exact sequence

$$(5) \qquad \begin{aligned} \cdots &\to \pi_{r-1}G(n-1) \overset{i_\#}{\to} \pi_{r-1}SG(n) \overset{p_\#}{\to} \pi_{r-1}(SG(n)/G(n-1)) \\ &\overset{\partial}{\to} \pi_{r-2}G(n-1) \overset{i_\#}{\to} \pi_{r-2}SG(n) \to \cdots \end{aligned}$$

of the fibration

$$(6) \qquad G(n-1) \overset{i}{\to} SG(n) \overset{p}{\to} SG(n)/G(n-1).$$

Recall that $\gamma \in \pi_{r-1}(SG(n)/G(n-1))$ classifies the reduction of the structure group of the trivial $SG(n)$ bundle ∂P_k to $G(n-1)$. Denote the reduced $G(n-1)$ bundle by $\partial P_k'$.

The bundle $\partial P_k'$ is classified by $\partial\gamma \in \pi_{r-2}G(n-1)$.

Let $j_\#\colon \pi_{r-2}G(n-2) \to \pi_{r-2}(G(n-1))$ be the natural inclusion.

CONSTRUCTION 4.1. *Let*

$$H_0 = \{\gamma \in \pi_{r-1}(SG(n)/G(n-1)) \colon \partial\gamma \in \operatorname{Im} j_\#\}.$$

Suppose $\gamma \in H_0$. $\partial P_k'$ is a $G(n-1)$ bundle over a sphere $S^{r-1} = \partial B_k(\Sigma)$. Remove a neighborhood U of a point x in the interior of the $(k-1)$ stratum of $\mathrm{cl}(\Sigma - N_k(\Sigma))$. Such a neighborhood is diffeomorphic to $D^{r-1} \times M_\rho$, where M_ρ is the mapping cylinder of $\rho\colon SG(n) \to SG(n)/G(n-1)$. Let V be the complement of U in $\Sigma - N_k(\Sigma)$.

$$(7) \qquad \begin{aligned} \partial\gamma\colon S^{r-2} &\to \{G(n) \text{ equivariant self-diffeomorphisms of } M_\rho\} \\ &= G(n-2) \times G(1) \hookrightarrow G(n-1). \end{aligned}$$

Note. After regluing U we may add back $N_k(\Sigma) \cong D^r \times SG(n)$ using the fact that $i_\#\partial\gamma = 0$. This completes the construction.

Denote the result of the Construction 4.1 by $\Sigma'_{(0,\partial\gamma)}$. If Σ has obstruction γ extending to $G(n)$ then clearly $\Sigma'_{(0,-\partial\gamma)}$ has obstruction γ' with $\partial\gamma' = 0$.

The construction can be carried out with $\beta \in \operatorname{Im}\partial \subseteq \pi_{r-1}(G(n-1))$ replacing $-\partial\gamma$. Denote the result by $\Sigma'_{0,\beta}$.

Let Σ be an $(n-1)$-axial $SG(n)$ manifold with an extension to an $(n-1)$-axial $G(n)$ action on $\mathrm{cl}(\Sigma - N_k(\Sigma))$.

Let $\gamma \in \pi_{r-1}(SG(n)/G(n-1))$ be the obstruction to extending the $G(n)$ action over Σ. Suppose $\partial\gamma = 0$ and $p_\#\alpha = \gamma$.

Define

$$\Sigma'_{(-\alpha,0)} = D^r \times SG(n) \cup_{G_\alpha} \mathrm{cl}(\Sigma - N_k(\Sigma)), \tag{8}$$

where

$$N_k(\Sigma) = D^r \times SG(n), \tag{9}$$

$$\partial\, \mathrm{cl}(\Sigma - N_k(\Sigma)) = \partial N_k(\Sigma) = S^{r-1} \times SG(n) \tag{10}$$

and

$$G_\alpha: S^{r-1} \times SG(n) \to S^{r-1} \times SG(n) \tag{11}$$

is defined by

$$G_\alpha(x, g) = \left(x, gf(x)^{-1}\right) \tag{12}$$

and $f: S^{r-1} \to SG(n)$ represents $\alpha \in \pi_{r-1}(SG(n))$:

LEMMA 4.2. *Let* Σ *be as above. Then*:

(i) *There is no obstruction to extending* $\Sigma'_{(-\alpha,0)}$ *to an* $(n-1)$*-axial* $G(n)$ *manifold.*

(ii) *The* $SG(n)$ *equivariant diffeomorphism class of* $\Sigma'_{(-\alpha,0)}$ *depends only on the class of* α mod Im $i_\#$, *and hence only depends on* γ *and* Σ.

PROOF. (i) The obstruction for extending $\Sigma'_{(-\alpha,0)}$ to a $G(n)$ action can easily be seen to be $p_\#(-\alpha) + \gamma = p_\#(-\alpha + \alpha) = 0$.

(ii) Let $\alpha' = \alpha \bmod \mathrm{Im}\, i_\#$ and let f' and f represent α' and α, respectively. If $\alpha' = \alpha$, then f' and f are homotopic smooth maps from ∂B_k to $SG(n)$. Choose a smooth homotopy $F: \partial B_k \times I \to SG(n)$ from $f(f')^{-1}$ to 1 where

$$f(f')^{-1}(b) = f(b) f'(b)^{-1} \tag{13}$$

and $1(b) = 1 \in SG(n)$. A diffeomorphism from $\Sigma'_{-\alpha,0}$ to $\Sigma'_{-\alpha',0}$ can be defined to be the identity on

$$\begin{aligned} \mathrm{cl}(\Sigma - N_k) &= \mathrm{cl}(\Sigma'_{-\alpha,0} - N_k(\Sigma'_{-\alpha,0})) \\ &= \mathrm{cl}(\Sigma'_{-\alpha',0} - N_k(\Sigma'_{-\alpha',0})), \end{aligned} \tag{14}$$

the identity on

$$B_k \times SG(n) - \{\text{collar of } \partial B_k \times SG(n)\} \tag{15}$$

and the homotopy F on the collar of $\partial B_k \times SG(n)$. This shows that the construction depends only on the homotopy class of α. We now show that $\Sigma'_{(-\alpha,0)} \cong \Sigma'_{(-\alpha',0)}$.

Case 1. Σ is the linear sphere, $p_\#\alpha = 0$, and $\alpha' = 0$. By exactness of the sequence (5), $\alpha \in \mathrm{Im}\, i_\#$, so that we may suppose that $f: \partial B_k \to G(k) \hookrightarrow SG(n)$ represents α. This means that G_α is $G(n)$ equivariant so that $\Sigma_{(-\alpha,0)}$ has a $G(n)$ action. We shall show that the ith twist invariant of $\Sigma_{(-\alpha,0)}$ is a homotopy equivalence for $i > 0$ and hence $\Sigma_{(-\alpha,0)}$ is a universal space (for $(n-1)$-axial $O(n)$ actions on homotopy spheres with the same dimension). By uniqueness of universal spaces (Theorem 1.5) it will follow that $\Sigma \cong \Sigma_{(-\alpha,0)}$ since Σ is the linear sphere by hypothesis.

Consider the diagram:

(16)

$$\begin{array}{ccccccc} P_i(\Sigma) & \leftrightarrow & \partial_0 P_{k-i}^{k-i} \times_{G(k-i)} P_i(\Sigma) & \overset{f^{i,k-i}}{\to} & \partial_i P_k(\Sigma) & & \overset{T}{\to} G(k) \\ & & & & & \searrow \cong & \uparrow \mathrm{proj}_2 \\ & & & & & & \partial B_k \times G(k) \\ \downarrow & & \downarrow & & & & \uparrow G_\alpha \\ & & & & & & \partial B_k \times G(k) \\ & & & & & \cong \nearrow & \downarrow \mathrm{proj}_2 \\ P_i(\Sigma'_{(-\alpha,0)}) & \hookrightarrow & \partial_0 P_{k-i}^{k-i} \times_{G(k-i)} P_i(\Sigma'_{(-\alpha,0)}) & \overset{f^{i,k-i}}{\to} & \partial_i P_k(\Sigma'_{(-\alpha,0)}) & & \overset{T}{\to} G(k) \end{array}$$

The top row represents the ith twist invariant of Σ and the bottom row represents the ith twist invariant of $\Sigma'_{(-\alpha,0)}$. Furthermore,

$$P_i(\Sigma) \cong P_i(\Sigma'_{(-\alpha,0)}) \cong S\left(H_+(i)^i \times \mathbf{R}^m\right) \times G(k) \tag{17}$$

and the maps in the top and bottom rows of the diagram are

$$(x, g) \to [h, x, g] \to \left[\begin{pmatrix} h & \\ & x \end{pmatrix}, g\right] \to g \tag{18}$$

and

$$(x, g) \to [h, x, g] \to \left[\begin{pmatrix} h & \\ & x \end{pmatrix}, f\begin{pmatrix} h & \\ & x \end{pmatrix} \cdot g\right] \to f\begin{pmatrix} h & \\ & x \end{pmatrix} \cdot g. \tag{19}$$

Both of these are homotopy equivalences for $i > 0$, since $S(H_+(i)^i \times \mathbf{R}^m)$ is contractible for $i > 0$. Now we have that Σ is equivariantly diffeomorphic to $\Sigma'_{(-\alpha,0)}$. Consider the classifying map for $\Sigma'_{(-\alpha,0)}$:

$$\begin{array}{ccc} \Sigma'_{(-\alpha,0)} & \to & \Sigma \\ \downarrow & & \downarrow \\ B & \overset{F_\#}{\to} & B \end{array} \tag{20}$$

By the Davis Classification Theorem (Theorem 1.4) for $(n-1)$-axial $O(n)$ actions the equivariant diffeomorphism type of $\Sigma'_{(-\alpha,0)}$ is determined by the isotopy class of $F_\#$ in the stratified maps $B \to B$, hence $F_\#$ is isotopic to the identity. Without loss of generality we can suppose that $F_\#$ is the identity.

Case 2. Arbitrary Σ, $\alpha' = 0$, $p_\# \alpha = 0$. Let S denote the linear sphere with the same dimension and dimension of fixed point sets as Σ. Consider the classifying map:

$$\begin{array}{ccc} \Sigma & \overset{F}{\to} & S \\ \downarrow & & \downarrow \\ B(\Sigma) & \overset{F_\#}{\to} & B(S) \end{array} \tag{21}$$

Applying the construction to both Σ and S we naturally get a map $F'_{-\alpha,0}$: $\Sigma'_{-\alpha,0} \to S'_{-\alpha,0}$, with $(F'_{-\alpha,0})_\# = F_\#$. The diagram

$$\begin{array}{ccccc} \Sigma'_{(-\alpha,0)} & \xrightarrow{F'_{(-\alpha,0)}} & S'_{(-\alpha,0)} & \xrightarrow{\cong} & S \\ \downarrow & & \downarrow & & \downarrow \\ B(\Sigma'_{(-\alpha,0)}) & \xrightarrow{F_\#} & B(S) & \xrightarrow{\text{id}} & B(S) \end{array} \tag{22}$$

shows that the classifying map for $\Sigma'_{(-\alpha,0)}$ is $F_\#$, hence Σ and $\Sigma'_{(-\alpha,0)}$ are equivariantly diffeomorphic.

Case 3. Arbitrary Σ, α, α' with $p_\#\alpha = p_\#\alpha'$. Then

$$\Sigma'_{(-\alpha',0)} \cong \left(\Sigma'_{(-\alpha',0)}\right)_{(-(-\alpha'+\alpha),0)} \cong \Sigma'_{(-\alpha,0)} \tag{23}$$

since $-\alpha' + \alpha$ is represented by $f'^{-1}f$ if f' represents α'. (Note that $(f')^{-1}(b) = [f'(b)]^{-1}$ represents $-\alpha'$ by the equivalence of the two group structures on $[\partial B_k, G(k)]$.) This completes the proof of the lemma.

DEFINITION 4.1.

$$\Sigma_{(-\alpha,-\partial\gamma)} = \left(\Sigma'_{(0,-\partial\gamma)}\right)'_{(-\alpha,0)} \tag{24}$$

and

$$\Sigma_{-\gamma} = \Sigma_{(-\alpha,-\partial\gamma)}. \tag{25}$$

We have proved that $\Sigma_{-\gamma}$ depends only on Σ and that $\Sigma_{-\gamma}$ admits an extension to $G(n)$.

REMARK 4.1. It can easily be seen that, for any $\gamma \in \pi_{r-1}(SG(n)/G(n-1))$, $\Sigma'_\gamma = \Sigma\#S'_\gamma$, where S denotes the unit sphere in $M(n, n-1) \times \mathbf{R}^m$ and $m - 1 = \dim F(\Sigma, SG(n))$. However, the definition of Σ'_γ above generalizes naturally to include $(n-1)$-axial $SG(n)$ actions without fixed points.

5. Classification of $(n-1)$-axial $SG(n)$ actions on homotopy spheres. Let $H = \pi_{r-1}SG(n)/G(n-1)$. If $SG(n) = SO(n)$, then let K be the trivial group. If $SG(n) = SU(n)$ let K be the homotopy theoretically defined group of section 3.2. Throughout this section we make the hypothesis

$$\begin{array}{lll} (m-1) \geqslant 1 & \text{if } n = 4 & \\ (m-1) \geqslant 1 & \text{if } n = 5 & \text{for } SG(n) = SO(n), \\ (m-1) \geqslant 2 & \text{if } n = 3 & \text{for } SG(n) = SU(n). \end{array} \tag{1}$$

Let $S_{\gamma,\sigma}$, with $\gamma \in H$, $\sigma \in K$ be the spaces defined in §4. Let r be the map of §3, (57).

THEOREM 5.1. *$(n-1)$-axial $SG(n)$ actions on homotopy spheres are classified by the following data.*

(i) *a choice of* $\gamma \in H$ (*modulo* Im r *if* $SG(n) = SU(n)$),

(ii) *a choice of* $\sigma \in K$ (*if* $SG(n) = SU(n)$),

(iii) *a stratified map*

$$F_\#: B \to B(S_{\gamma,\sigma}) \tag{2}$$

of orbit spaces such that $F_{\#}$ induces integral homology isomorphisms

(3) $$F_{\#,i}\colon B_i \to B_i(S_{\gamma,\sigma}), \qquad 0 < i \leqslant n-1,$$

if $SG(n) = SU(n)$ or integral homology isomorphisms

(4) $$F_{\#,i}\colon D_i \to D_i(S_{\gamma,\sigma}) \quad \textit{for } n \equiv i \bmod 2$$

if $SG(n) = SO(n)$.

Here D_i denotes the ith double branched cover of the B induced (by pullback) from $P_i(S_{\gamma,\sigma})$.

The $SG(n)$ homotopy sphere corresponding to the data $F_{\#}$ and S_γ is given by the pullback:

(5) $$\begin{array}{ccc} \Sigma & \to & S_{\gamma,\sigma} \\ \downarrow & & \downarrow \\ B & \overset{F_{\#}}{\to} & B(S_{\gamma,\sigma}) \end{array}$$

Conversely given an $SG(n)$ homotopy sphere Σ over B there is a $\gamma \in H$ and $\sigma \in K$ such that there exists an equivariant stratified map

(6) $$F\colon \Sigma \to S_{\gamma,\sigma}.$$

Furthermore, F is unique up to isotopy through equivariant stratified maps. Σ then corresponds to the data $F_{\#}$, γ and σ.

PROOF. Given the data of (i)–(iii), Σ is an $SG(n)$ homotopy sphere by (ii), (3) or (4) and the Homology Isomorphism Theorem (Theorem 1.3).

Conversely, given Σ as above, let σ, γ be the obstructions to extending Σ to $U(n)$. (σ is trivial if $SG(n) = SO(n)$; if $SG(n) = SU(n)$, then γ is defined after killing σ—the inverse of realization—then extending the action over $\mathrm{cl}(\Sigma - N_k(\Sigma))$ in a standard way, depending on σ and not on Σ.) Then γ is killed by the constructions of §4, giving $\Sigma_{-\gamma,-\sigma}$ which admits a $G(n)$ action which is unique up to $G(n)$ equivariant diffeomorphism by Theorem 4.2. By the Davis Classification Theorem (Theorem 1.4) there exists a unique $G(n)$ equivariant stratified map

(7) $$F\colon \Sigma_{-\gamma,-\sigma} \to S.$$

Performing the realization of γ, σ to this map gives

(8) $$F_{\gamma,\sigma}\colon \Sigma \to S_{\gamma,\sigma}.$$

This map is unique since its existence implies that $\Sigma'_{-\gamma,-\sigma}$ was extended to $G(n)$ in the same standard way as S was. If $SG(n) = SO(n)$ recall that $\sigma = 1$ and γ is unique. If $SG(n) = SU(n)$ then σ is unique and γ has indeterminacy $\mathrm{Im}\, r$ by Lemma 4.1.

6. The classifying spaces S_α. Let S be the unit sphere in $M(n, n-1) \times \mathbf{R}^m$.

THEOREM 6.1. *The classifying spaces S_α with $\alpha \in \mathrm{Im}\, p_{\#}$ (see the sequence (1) of §4) bound parallelizable manifolds and with the dimension assumptions $n \geqslant 3$ and $m > 2n$, if $SG(n) = SU(n)$, $n \geqslant 4$ if $SG(n) = SO(n)$ and $m - 1$ $(= \dim F(S_\alpha, SG(n))) \geqslant 2$, the spaces S_α have standard differential structure.*

PROOF. Let

(1) $$f\colon S^{r-1} \to SG(n)$$

represent $\alpha \in \pi_{r-1}(SG(n))$, and let

(2) $$F\colon S^{r-1} \times SG(n) \to S^{r-1} \times SG(n)$$

be defined by

(3) $$F(x, g) = \left(x, gf(x)^{-1}\right).$$

Let

(4) $$D^r \times SG(n) \hookrightarrow S$$

be a tubular neighborhood of a principal orbit.

Then

(5) $$S_\alpha = S - \operatorname{Int}(D^r \times SG(n)) \cup_F D^r \times SG(n).$$

Define

(6) $$\tilde{F}\colon S^{r-1} \times SG(n) \to S^{r-1} \times SG(n)$$

by

(7) $$\tilde{F}(x, g) = (x, f(x)g).$$

Then $\tilde{F}$ extends to a map

(8) $$\tilde{F}\colon S^{r-1} \times M_\rho \to S^{r-1} \times M_\rho,$$

given by

(9) $$\tilde{F}(x, [g, t]) = (x, [f(x)g, t]),$$

where M_ρ denotes the mapping cylinder of the projection $\rho\colon SG(n) \to SG(n)/SG(2)$, $x \in S^{r-1}$, $g \in SG(n)$, $t \in I$ and (g, t) represents a point of M_ρ.

Let D be the unit disc in $M(n, n-1) \times \mathbf{R}^m$, let $\pi\colon D \to B(D)$ denote the orbit map. Let T be a tubular neighborhood of a curve in $B(D)$ from a point in $\operatorname{Int}(B_k(S))$, through principal orbits to $B_{k-1}(D)$. Then $\pi^{-1}(T) \cong D^r \times M_\rho$.

Let $\tilde{S}_\alpha$ be obtained from S using $\tilde{F}$ in the same way as S_α was obtained from S using F. Let $\tilde{D}_\alpha$ be similarly obtained from D using $\tilde{F}\colon S^{r-1} \times M_\rho \to S^{r-1} \times M_\rho$. Then $\tilde{D}_\alpha$ naturally bounds $\tilde{S}_\alpha$. Furthermore by Mayer-Vietoris sequences (or by an argument using the classifying maps of D, $\tilde{D}_\alpha$ and the Homology Isomorphism Theorem) $\tilde{D}_\alpha$ is acyclic. It is easily seen that $\tilde{D}_\alpha$ is simply connected, hence $\tilde{S}_\alpha$ bounds a contractible manifold.

To show $S_\alpha \in bP_{N+1}$ ($N = \dim S$) it suffices to show that $\tilde{S}_\alpha$ and S_α represent the same element of coker J under the Pontryagin-Thom construction.

Let

(10) $$i\colon SG(n) \hookrightarrow S$$

be the inclusion of a principal orbit. The natural left invariant framing of $M(n, n-1) \times \mathbf{R}^m$ extends the framing of $D^r \times SG(n)$ which extends the left invariant framing of $SG(n)$. We define a new embedding $j\colon SG(n) \to S$ defined by making the diagram

$$
\begin{array}{ccccc}
 & & \overset{j}{\frown} & & \\
SG(n) & \overset{\text{Inv}}{\to} & SG(n) & \overset{i}{\to} & S \\
\| & & \| & & \\
X_1 & & X_2 & &
\end{array}
\tag{11}
$$

commute, where $\mathrm{Inv}(g) = g^{-1}$ is the inverse map on $SG(n)$.

Let $(X_1, \mathscr{F}_1)$, $(X_2, \mathscr{F}_2)$ denote the two copies of $SG(n)$, let $\mathscr{F}_1$ be the left invariant framing on $X_1 = SG(n)$, and let $\mathscr{F}_2$ be the right invariant framing on $X_2 = SG(n)$.

$$\mathrm{Inv}\colon (X_1, \mathscr{F}_1) \to (X_2, \mathscr{F}_2) \tag{12}$$

is then a diffeomorphism of framed manifolds. Since j is homotopic to i (by obstruction theory), the left invariant framing $\mathscr{F}_1$ of $SG(n)$ extends via j to S. This means that $\mathscr{F}_2$ extends via i to a framing $\mathscr{F}_2$ of S.

Using the framings $\mathscr{F}_1$ (resp. $\mathscr{F}_2$) on S we see that the difference in framed cobordism between $\tilde{S}_\alpha$ and S (resp. S_α and S) is represented by a principal $SG(n)$ bundle

$$SG(n) \to \tilde{E}_\alpha \to S^r \tag{13}$$

$$(\text{resp. } SG(n) \to E_\alpha \to S^r), \tag{14}$$

where the structure group acts on the left by left multiplication (resp. on the right by the inverse of right multiplication), and the clutching function is given by the map f of (1). But the natural map Inv: $SG(n) \to SG(n)$ induces a diffeomorphism

$$\mathrm{Inv}\colon E_\alpha \to \tilde{E}_\alpha \tag{15}$$

of framed manifolds. This proves that $S_\alpha \in bP_{N+1}$.

Now suppose that the dimension assumptions of the theorem will hold. Choose a homotopy

$$K\colon SG(n) \times I \to S^N \times I \tag{16}$$

from i to j, which is smooth and conditioned, i.e.

$$K|_{SG(n) \times [0,\varepsilon)} = (i \times \mathrm{id})|_{[0,\varepsilon]} \tag{17}$$

and

$$K|_{SG(n) \times [1-\varepsilon,1]} = (j \times \mathrm{id})|_{[1-\varepsilon,1]}, \tag{18}$$

for some $\varepsilon > 0$.

The dimension hypothesis guarantees that $2\dim(SG(n) \times I) < \dim S^N \times I$. By general position, K can be approximated rel $SG(n) \times (\Sigma 0, \varepsilon] \cup [1-\varepsilon, 1])$ to an embedding $\overline{K}$. We can and do suppose that $P_1 \circ \overline{K}$ is an immersion, where P_1: $S^n \times I \to S^n$ is projection on the first factor, and that $\mathrm{Im}(P_1 \circ \overline{K})$ has only finitely many self-intersection points with different t values, i.e.

$$\overline{K}(g, t) = \overline{K}(g', t') \Rightarrow t \neq t'. \tag{19}$$

Let

(20) $$\phi = \left(P_1 \circ \overline{K}\right) \times \mathrm{id}\colon SG(n) \times I \to S \times I.$$

$\overline{K}$ is smooth although ϕ may not be. However, it follows from (19) that ϕ is a homeomorphism. Let Γ be the smooth structure on $SG(n) \times I$ induced by ϕ. Let Γ_0 denote the usual structure on $SG(n)$. Then by a "Concordance Implies Isotopy" theorem (see Kirby and Siebenmann [**16**, p. 25]), there is an isotopy h_t of the identity of $SG(n) \times I$ rel $SG(n) \times \{0\}$ and preserving $SG(n) \times \{1\}$, so that

(21) $$h_1^*(\Gamma) = \Gamma_0 \times I.$$

Define an isotopy

(22) $$H(g,t) = \begin{cases} h_{2t}(g,t), & 0 \leqslant t \leqslant \frac{1}{2}, \\ h_1(g, 1-t), & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Then $\phi \circ H$ is a smooth isotopy of imbeddings from i to j. Since $\tilde{S}_\alpha$ is obtained from S using i in exactly the same way as S_α is obtained from S using j, the deformation (21) above gives a deformation between the constructions of $\tilde{S}_\alpha$ and S_α, hence the differential structures $\tilde{S}_\alpha$ and S_α are isomorphic. Since $\tilde{S}_\alpha$ has standard differential structure, this completes the proof.

7. Biaxial actions on homotopy 7-spheres. We reinterpret the results of Davis [**10**].

Let $n = 3$, $m = 2$ and $SG(n) = SO(3)$, so that $r = 4$. In the sequence (5) of §4 the map

(1) $$i_\#\colon \pi_3(SO(3)) \to \pi_3(SO(3)/O(2))$$

is an isomorphism onto, so that the set of classifying spaces for biaxial $SO(3)$ actions on homotopy 7-spheres is

(2) $$\{S_{k\alpha}\colon k \in \mathbf{Z}\},$$

where α denotes a generator of Im $i_\#$. Consider the classifying map

(3) $$\Sigma^7_{2,-1} \to S_{k\alpha}.$$

Performing the construction of §4 in reverse gives an $O(3)$ equivariant stratified map

(4) $$F\colon \Sigma^7_{2,-1} \# S_{-k\alpha} \to S.$$

Since $B(S_{k\alpha}) = B(S)$ and $B(\Sigma^7_{2,-1}) = B(S)$, $B(\Sigma^7_{2,-1} \# S_{-k\alpha}) = B(S)$, and since biaxial $O(3)$ actions are classified by their orbit spaces, F must be an equivariant diffeomorphism. We obtain

(5) $$\Sigma^7_{2,-1} \underset{SO(3)}{\approx} S_{k\alpha}.$$

Since by Davis [**10**] $\Sigma^7_{2,-1}$ generates the group of biaxial $SO(3)$ homotopy spheres over $B(S)$ we must have $k = \pm 1$; without loss of generality $k = 1$. We have shown that

(6) $$\Sigma^7_{2,-1} \underset{SO(3)}{\cong} S_\alpha.$$

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